Journal of Statistical Physics, Vol. 21, No. 6, 1979

The Distribution of Clusters for the Ising Model

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Received May 17, 1979

We rigorously prove that the probability P_n for the origin to belong to a cluster of exactly n positive spins in the ν-dimensional Ising model behaves as $\exp(-\alpha n^{(\nu-1)/\nu})$ in various regions, including in particular the low-temperature positive and negative phases in zero magnetic field.

KEY WORDS: Ising model; percolation; cluster size distribution; FKG inequalities.

1. INTRODUCTION

The existence of two different behaviors of the cluster size distribution function in the random percolation problem above and below the critical concentration has been proved by Kunz and Souillard,[1] who obtained moreover for low and large concentrations the exact behavior of the cluster size distribution. They proved also that for interacting percolation problems, this distribution cannot decay exponentially in the percolative region, whereas it does in the low-concentration region.

In this paper we shall study the case of the ferromagnetic ν-dimensional Ising model and obtain the exact behavior of the cluster distribution function in various regions of interest. In order to state our results, we first introduce some definitions.

We consider the ν-dimensional cubic lattice $\mathbb{Z}^\nu$ ($\nu \geqslant 2$) and boxes $\Lambda \subset \mathbb{Z}^\nu$. The Ising model is usually defined by the energy associated with a spin configuration σ_Λ in Λ:

$$U(\sigma_\Lambda) = \sum_{x\in\Lambda} -h\sigma_x - \beta \sum_{\langle x,y\rangle} \sigma_x\sigma_y$$

where σ_x denotes the spin at the site x of the lattice; σ_x can take the values $\pm 1/2$. The second summation runs over all pairs $\langle x, y\rangle$ of nearest neighbor

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sites, and β denotes the reciprocal temperature (including the constant of the interaction). Finally, possible boundary terms would also appear in the case of boundary conditions.

In this paper, it will be useful to work with the variables $\pi_x = \sigma_x + \frac{1}{2}$. If $\pi_x = 1$, we shall say that the site x is occupied and if $\pi_x = 0$, it will be said to be empty. Then, up to an additive constant, the energy of a configuration π_Λ in the box Λ can be written as

$$U(\pi_\Lambda) = -h \sum_{x \in \Lambda} \pi_x + \beta \sum_{\langle x, y \rangle} \pi_x(1 - \pi_y)$$

or, if X denotes the set of occupied sites in Λ,

$$U(X) = -h|X| + \beta S_\Lambda(X)$$

where $|X|$ denotes the number of sites in the set X and

$$S_\Lambda(X) = \sum_{\substack{x \in X \\ y \in \Lambda \setminus X \\ \langle x, y \rangle}} 1$$

Then the probability of the configuration in Λ where the sites of X are occupied and the sites of $\Lambda \setminus X$ are empty is by definition

$$\mu_\Lambda(X) = e^{-U(X)}/Z_\Lambda, \qquad \text{with} \qquad Z_\Lambda = \sum_{Y \subset \Lambda} e^{-U(Y)}$$

Finally, the Ising model is obtained by taking the thermodynamic limit of the probability measure defined above, that is, Λ grows to $\mathbb{Z}^\nu$ in a sufficiently regular way.

A cluster C is a maximal connected set of occupied sites (plus spins), i.e., a set of occupied sites which are connected through the bonds of the lattice and are surrounded by empty sites. We call ∂C its boundary, that is, the set of empty sites that are nearest neighbor to the sites of C.

We introduce now the probability P_n for the origin to belong to a cluster of exactly n sites, so we can write it as

$$P_n = \sum_{\substack{0 \in C \\ |C| = n}} P(C, \partial C) = \sum_{\substack{0 \in C \\ |C| = n}} \bar{P}(C)$$

where $P(C, \overline{\partial C}) = \bar{P}(C)$ is the probability of occurrence of the cluster C, that is, the probability that C is occupied and ∂C empty. We call P_∞ the probability that the origin belongs to an infinite cluster of occupied sites.

It is known that:

1. The percolative region, i.e., the region such that $P_\infty > 0$, includes[2] the following regions: (a) sufficiently large, positive magnetic field for all temperatures; (b) arbitrary, positive field and $T < T_1$ for some temperature T_1; (c) $T < T_1$ in the positively magnetized pure phase in zero external field.

2. In the percolative regions, the P_n do not decay exponentially (in

contrast with the region with large, negative magnetic field). More precisely, it has been proved[1] that the moments of the cluster distribution function satisfy

$$\langle |C|^l \rangle \geqslant K^l \left(\frac{\nu}{\nu - 1} l \right)!$$

If $P_n \sim \exp(-\alpha n^{\xi})$, then this implies $\xi \leqslant (\nu - 1)/\nu$; $\xi = (\nu - 1)/\nu$ is a behavior proposed in Ref. 3 on the basis of numerical analysis in the low-temperature Ising model.

In this paper, we prove the following result:

Theorem. The cluster size distribution function for plus spins satisfies

$$\exp[-\alpha'(T, h) n^{(\nu-1)/\nu}] \leqslant P_n \leqslant \exp[-\alpha(T, h) n^{(\nu-1)/\nu}]$$

in the four following regions:

(1) Large magnetic field, $h - 2\nu/T \geqslant h_0$.
(2) Positive magnetic field and low temperatures, $h > 0$ and $T < T_0$.
(3) Zero magnetic field and low temperature in the positive phase, $h = 0^+$ and $T < T_0$.
(4) Zero magnetic field and low temperature in the negative phase, $h = 0^-$ and $T < T_0$.

We neglect for the moment case 4 of the negative phase at $h = 0$ and we first prove the theorem in the other three cases.

2. PRELIMINARIES

In order to be self-consistent, we briefly indicate the proof of the following lemma, which is the basis of the proof of the analogous result in the random percolation case by Kunz and Souillard,[1] and then we will turn to the proof of the three hypotheses of the lemma in our situations.

Lemma 1. Let us suppose that for all m and n

$$P_{m+n}/(m + n) \geqslant (P_m/m)P_n/n \tag{1}$$

$$P_n/n \leqslant \exp(-\alpha n^{(\nu-1)/\nu}) \tag{2}$$

$$Q_n = \sum_{t=n}^{\infty} (P_t/t) \geqslant \exp(-\delta n^{(\nu-1)/\nu}) \tag{3}$$

Then there exists a real, positive α' such that

$$P_n/n \geqslant \exp(-\alpha' n^{(\nu-1)/\nu}) \tag{4}$$

Proof. The inequalities (2) and (3) imply that there exist an integer A and a real δ' such that

$$Q_n - Q_{An} \geqslant \exp(-\delta' n^{(\nu-1)/\nu}) \tag{5}$$

It is easy to see that (5) implies that we can find a positive δ'' such that in all the interval $[n, An[$ there exists an integer k satisfying

$$P_k/k \geqslant \exp(-\delta'' k^{(\nu-1)/\nu}) \tag{6}$$

If we consider now the intervals $I_i = [A^i, A^{i+1}[$, we denote by k_i, for each i, one of the integers of the interval I_i for which (6) holds. Now for each integer n we can make the "division on the basis $\{k_i\}$"; i.e., if n is in I_{l+1}, there exist integers $a_i(n)$ satisfying the following properties:

$$\begin{gathered} n = \sum_{i=1}^{l} a_i(n) k_i + a_0(n) \\ 0 \leqslant a_i < A^2, \qquad 0 \leqslant a_0 < A \end{gathered} \tag{7}$$

Then (1) and (6) together imply that

$$\frac{P_n}{n} \geqslant \prod_{i=1}^{l} [\exp(-\delta'' k_i^{(\nu-1)/\nu})]^{a_i} \frac{P_{a_0}}{a_0} \tag{8}$$

Now if we notice that P_{a_0}/a_0 is bounded from below for a_0 belonging to $[1, A[$ and that there exists a constant C such that

$$\sum_{i=1}^{l} a_i(n) k_i^{(\nu-1)/\nu} \leqslant C n^{(\nu-1)/\nu} \tag{9}$$

we obtain the announced result for all n:

$$P_n/n \geqslant \exp(-\alpha' n^{(\nu-1)/\nu})$$

Let us now mention the FKG inequalities,[4] which will be very useful in the following. A function f defined on the configurations of a lattice is said to be increasing if $f(X) \leqslant f(Y)$ when $X \subset Y$ and decreasing if $f(X) \geqslant f(Y)$ when $X \subset Y$. In the same way, we will say that an event is increasing (resp. decreasing) if its characteristic function is increasing (resp. decreasing). Now let μ be a probability measure over the configurations satisfying, as in the Ising model,

$$\mu(X \cup Y)\mu(X \cap Y) \geqslant \mu(X)\mu(Y)$$

Then the FKG inequalities tell us that $\langle f \cdot g \rangle \geqslant \langle f \rangle \langle g \rangle$ whenever f and g are both increasing (or decreasing) functions and consequently $\langle f \cdot g \rangle \leqslant \langle f \rangle \langle g \rangle$ if one of them is decreasing and the other increasing.

Now we can begin the proof of the three hypotheses of Lemma 1 in our situations.

3. PROOF OF THE BASIC INEQUALITIES

Proposition 1. For every β and h, and for any possible phase, the P_n satisfy

$$P_{n+m}/(n+m) \geqslant (P_m/m) P_n/n$$

Proof. We will first prove that for two disjoint clusters C_1 and C_2

$$\bar{P}(C_1 \cup C_2) \geqslant \bar{P}(C_1)\bar{P}(C_2)$$

Consider a box Λ, with given boundary conditions and a cluster C strictly included in it, i.e., such that $\partial C \cap \partial\Lambda = \varnothing$. Then

$$\bar{P}_\Lambda(C) = P_\Lambda(C, \overline{\partial C}) = \sum_{Y \subset \Lambda\setminus C \cup \partial C} \frac{e^{-U(C \cup Y)}}{Z_\Lambda} \tag{10}$$

Since the interaction is only between nearest neighbors, since $Y \subset \Lambda/C \cup \partial C$, we have

$$U(C \cup Y) = U(C) + U(Y) \tag{11}$$

Then (10) becomes

$$\begin{aligned}\bar{P}_\Lambda(C) &= e^{-U(C)} \sum_{Y \subset \Lambda\setminus C \cup \partial C} \frac{e^{-U(Y)}}{Z_\Lambda} \\ &= e^{-U(C)} \langle \tilde{\pi}_{C \cup \partial C} \rangle_\Lambda \\ &= e^{-U(C)} P_\Lambda(C \cup \partial C)\end{aligned} \tag{12}$$

where $\tilde{\pi}_X$ is the characteristic function of the event "X is empty", and $P(\bar{X})$ denotes the probability for the X to be empty. Now let us consider two clusters C_1 and C_2 with $C_1 \cap C_2 = \varnothing$ and let $C_1 \cup C_2 = C$. Then because the interaction is positive and between nearest neighbors,

$$U(C_1 \cup C_2) \leqslant U(C_1) + U(C_2)$$

that is,

$$e^{-U(C_1)} e^{-U(C_2)} \leqslant e^{-U(C)} \tag{13}$$

and applying FKG inequalities to both decreasing functions $\tilde{\pi}_{C_1 \cup \partial C_1}$ and $\tilde{\pi}_{C_2 \cup \partial C_2}$, we have

$$\langle \tilde{\pi}_{C_1 \cup \partial C_1} \rangle_\Lambda \langle \tilde{\pi}_{C_2 \cup \partial C_2} \rangle_\Lambda \leqslant \langle \tilde{\pi}_{C_1 \cup \partial C_1} \tilde{\pi}_{C_2 \cup \partial C_2} \rangle_\Lambda \leqslant \langle \tilde{\pi}_{C_1 \cup \partial C_1 \cup C_2 \cup \partial C_2} \rangle_\Lambda = \langle \tilde{\pi}_{C \cup \partial C} \rangle_\Lambda \tag{14}$$

So now using (12)–(14), we obtain

$$\begin{aligned}\bar{P}_\Lambda(C_1)\bar{P}_\Lambda(C_2) &= e^{-U(C_1)} \langle \tilde{\pi}_{C_1 \cup \partial C_1} \rangle_\Lambda e^{-U(C_2)} \langle \tilde{\pi}_{C_2 \cup \partial C_2} \rangle_\Lambda \\ &\leqslant e^{-U(C)} \langle \tilde{\pi}_{C \cup \partial C} \rangle_\Lambda = \bar{P}_\Lambda(C)\end{aligned}$$

In turn this ensures that after the thermodynamic limit and for any phase

$$\bar{P}(C_1)\bar{P}(C_2) \leqslant \bar{P}(C_1 \cup C_2) \tag{15}$$

Now the end of the proof follows as in Ref. 1. We restrict ourselves to $\mathbb{Z}^2$ for the sake of simplicity. We can write P_n/n as $\sum_{|C'|=n} \bar{P}(C')$, where the summation runs now only on the different shapes of clusters of size n.

To each couple (C_1', C_2') we can associate the shape $C' = C_1' + C_2'$ by translating C_2' in such a way that the lowest among the points of C_2' farther on the left becomes the nearest neighbor on the right of the highest among the points in C_1' farther on the right. It is easy to see that, in this way, we have defined an injection from $\{C_1'\}_{n_1} \times \{C_2'\}_{n_2}$ into $\{C'\}_{n_1+n_2}$, where $\{C'\}_n$ denotes the set of different shapes of clusters of size n. Now, using (15), we can write

$$P_{n_1+n_2}/(n_1 + n_2) = \sum_{|C'|=n_1+n_2} \bar{P}(C') \geqslant \sum_{|C_1'|=n_1} \sum_{|C_2'|=n_2} \bar{P}(C_1' + C_2') \tag{16}$$

$$\geqslant \sum_{|C_1'|=n_1} \sum_{|C_2'|=n_2} \bar{P}(C_1')\bar{P}(C_2') = (P_{n_1}/n_1)P_{n_2}/n_2 \tag{17}$$

This achieves the proof of our first proposition.

Proposition 2. There exists a positive, real α depending on β and h, such that the P_n satisfy

$$P_n/n \leqslant \exp(-\alpha n^{(\nu-1)/\nu})$$

in the following three regions:

(1) Large magnetic field, $h - 2\nu\beta \geqslant h$.
(2) Low temperature, and positive magnetic field, $T < T_0$ and $h > 0$.
(3) $T < T_0$ and $h = 0$ in the positive phase.

Proof. For each cluster C we will denote $\partial^e C$ as its external boundary, that is, the set of points in ∂C that is "linked to infinity" by a path in $\mathbb{Z}^\nu \backslash C$. This definition implies:

(i) $\partial^e C$ is a *connected set; we mark with a star the properties relative to the *lattice, that is, the lattice obtained from $\mathbb{Z}^\nu$ by adding the diagonals of all the elementary squares.

(ii) This set $\partial^e C$ divides $\mathbb{Z}^\nu$ into two parts: one internal and the other external.

(iii) $\partial^e C$ is a minimal set satisfying (i) and (ii).

Let us remark that if we denote by $\mathscr{C}$ a possible external boundary of a cluster C, and if $|C| = n$, then necessarily $|\mathscr{C}| \geqslant rn^{(\nu-1)/\nu}$, where r is a constant depending on the lattice. In all the following, summations over $\mathscr{C}$ will run over possible shapes of an external boundary of a cluster, and $\Delta\mathscr{C}$ will denote the set of points internal to $\mathscr{C}$ and nearest neighbor to some point of $\mathscr{C}$. Thus

$$P_n/n \leqslant \sum_{\mathscr{C}:|\mathscr{C}| \geqslant rn^{(\nu-1)/\nu}} P(\bar{\mathscr{C}}, \Delta\mathscr{C}) \tag{18}$$

Now let us suppose that we can find a number $q = q(h, \beta)$ such that

$$P(\bar{\mathscr{C}}, \Delta\mathscr{C}) \leqslant q^{|\mathscr{C}|} \tag{19}$$

and $q(h, \beta)$ is as small as we want in some regions of the plane (h, β); a Peierls estimate, applied to the *lattice, tells us that the number of such contours, $|\mathscr{C}|$ being fixed, is smaller than $K^{|\mathscr{C}|}$, where K is a constant depending on the *lattice. Then we would have

$$P_n/n \leqslant \sum_{k \geqslant rn^{(\nu-1)/\nu}} K^k q^k \tag{20}$$

$$\leqslant K'^{n^{(\nu-1)/\nu}} \tag{21}$$

for q sufficiently small, and K' goes to zero as q goes to zero. Inequality (21) is just the desired upper bound and it remains for us to find the upper bound (19) in the various regions mentioned in Proposition 2.

(a) Upper Bound for Large Magnetic Field

Clearly $P_\Lambda(\bar{\mathscr{C}}, \Delta\mathscr{C})$ is smaller than $P_\Lambda(\bar{\mathscr{C}})$, which we can write as

$$P_\Lambda(\bar{\mathscr{C}}) = \sum_{X \subset \Lambda \setminus \mathscr{C}} e^{-U(X)} \Big/ \sum_{\substack{X \subset \Lambda \setminus \mathscr{C} \\ Y \subset \mathscr{C}}} e^{-U(X \cup Y)} \tag{22}$$

By virtue of (13) we have again

$$e^{-U(X \cup Y)} \geqslant e^{-U(X)} e^{-U(Y)}$$

Furthermore, if we notice that $e^{-U(Y)}$ is always bounded from below by $e^{(h-2\nu\beta)|Y|}$, since a site has at most 2ν nearest neighbors, we obtain

$$\sum_{Y \subset \mathscr{C}} e^{-U(Y)} \geqslant \sum_{|Y|=0}^{|\mathscr{C}|} \frac{|\mathscr{C}|!}{|Y|!\,(|\mathscr{C}| - |Y|)!} e^{(h-2\nu\beta)|Y|} \tag{23}$$

Then (22) yields that at the thermodynamic limit

$$P(\bar{\mathscr{C}}) \leqslant \left(\frac{1}{1 + e^{(h-2\nu\beta)}}\right)^{|\mathscr{C}|} \tag{24}$$

which is a suitable upper bound for sufficiently large magnetic field.

(b) Upper Bound for Low Temperatures

First, let us restrict ourselves to $h = 0$ and look at the positively magnetized phase. We choose as usual a box Λ with the following boundary conditions: every point of $\partial\Lambda$ is occupied. If $\mathscr{C}$ is a contour strictly included in the box Λ, we have to bound the conditional probability $P_\Lambda(\Delta\mathscr{C}, \bar{\mathscr{C}} | \partial\Lambda)$, which we can write as

$$P_\Lambda(\Delta\mathscr{C}, \bar{\mathscr{C}} | \partial\Lambda) = \frac{P_{\Lambda'}(\Delta\mathscr{C}, \bar{\mathscr{C}}, \partial\Lambda)}{P_{\Lambda'}(\partial\Lambda)} \tag{25}$$

$$= \frac{P_{\Lambda'}(\partial\Lambda | \Delta\mathscr{C}, \bar{\mathscr{C}})}{P_{\Lambda'}(\partial\Lambda)} P_{\Lambda'}(\Delta\mathscr{C}, \bar{\mathscr{C}}) \tag{26}$$

where $\Lambda' = \Lambda \cup \partial\Lambda$. But the Markovian property of the model implies that $P_{\Lambda'}(\partial\Lambda | \Delta\mathscr{C}, \bar{\mathscr{C}}) = P_{\Lambda'}(\partial\Lambda | \bar{\mathscr{C}})$, and now applying FKG inequalities to the functions $\bar{\pi}_{\mathscr{C}}$ and $\pi_{\partial\Lambda}$, which are respectively decreasing and increasing, we have

$$\frac{P_{\Lambda'}(\partial\Lambda | \bar{\mathscr{C}})}{P_{\Lambda'}(\partial\Lambda)} = \frac{\langle \bar{\pi}_{\mathscr{C}} \pi_{\partial\Lambda} \rangle_{\Lambda'}}{\langle \pi_{\partial\Lambda} \rangle_{\Lambda'} \langle \bar{\pi}_{\mathscr{C}} \rangle_{\Lambda'}} \leqslant 1 \tag{27}$$

Thus $P_\Lambda(\bar{\mathscr{C}}, \Delta\mathscr{C} | \partial\Lambda)$ is now bounded by $P_{\Lambda'}(\bar{\mathscr{C}}, \Delta\mathscr{C})$, which in turn ensures that

$$P_\Lambda(\bar{\mathscr{C}}, \Delta\mathscr{C} | \partial\Lambda) \leqslant P_{\Lambda'}(\bar{\mathscr{C}} | \Delta\mathscr{C}) \tag{28}$$

Now the well-known Peierls argument ensures the following bounds:

$$P_{\Lambda'}(\bar{\mathscr{C}} | \Delta\mathscr{C}) \leqslant e^{-\beta|\mathscr{C}|} \tag{29}$$

Inequality (29) is conserved after the thermodynamic limit, which ends the proof of (19) for $h = 0$ in the positively magnetized phase and $T < T_0$ for some T_0.

Now, to extend the inequality (29) to the second region, we will observe that $P(\bar{\mathscr{C}} | \Delta\mathscr{C})$ is a decreasing function on h. Since the conditional probability is again a probability satisfying the condition for the FKG inequalities and since

$$dP(\bar{\mathscr{C}} | \Delta\mathscr{C})/dh = \sum_{x \in \mathbb{Z}^\nu} \{P(x, \bar{\mathscr{C}} | \Delta\mathscr{C}) - P(x | \Delta\mathscr{C}) P(\bar{\mathscr{C}} | \Delta\mathscr{C})\} \tag{30}$$

$$= \sum_{x \in \mathbb{Z}^\nu} \{\langle \pi_x \bar{\pi}_{\mathscr{C}} \rangle_{|\Delta\mathscr{C}} - \langle \pi_x \rangle_{|\Delta\mathscr{C}} \langle \bar{\pi}_{\mathscr{C}} \rangle_{|\Delta\mathscr{C}}\} \tag{31}$$

where $\langle f \rangle_{|\Delta\mathscr{C}}$ denotes the average of f with the conditional measure, we have

$$dP(\bar{\mathscr{C}} | \Delta\mathscr{C})/dh \leqslant 0 \tag{32}$$

which ends the proof of the second proposition for $h > 0$ and $T < T_0$.

Proposition 3. In the regions 1–3, the Q_n decrease at most as $\exp(-\alpha' n^{(\nu-1)/\nu})$.

Let us consider the cubes in $\mathbb{Z}^\nu$, centered at the origin. We denote by γ such a cube and, as previously, by $\partial\gamma$ its boundary, by $\Delta\gamma$ the sites of γ nearest neighbor to some point of $\partial\gamma$, and by $\theta(\gamma)$ the complement of $\Delta\gamma$ in γ.

Now in P_n/n we will keep only the contribution of the clusters whose external boundary is a cubic one. These clusters are then composed of the points in $\Delta\gamma$ which have to be occupied and by exactly $n - \Delta\gamma$ points in $\theta(\gamma)$ which are both occupied and connected to $\Delta\gamma$. If we call $\chi_x^{\Delta\gamma}$ the characteristic function of the event "the point x in $\theta(\gamma)$ is connected to $\Delta\gamma$", a lower bound for P_n/n is then given by

$$P_n/n \geqslant \sum_{\gamma} P\left(\Delta\gamma, \partial\bar{\gamma}, \sum_{x\in\theta(\gamma)} \chi_x^{\Delta\gamma} = n - \Delta\gamma\right) \tag{33}$$

Moreover, we know [see Ref. 1, Eq. (51)] that for any positive function g depending only on the configurations inside $\theta(\gamma)$ there exists $B(h, \beta)$ for any $h \geqslant 0$ such that

$$\langle g\bar{\pi}_{\partial\gamma}\pi_{\Delta\gamma}\rangle \geqslant \langle g\rangle B^{|\partial\gamma|} \tag{34}$$

Thus (33) becomes

$$P_n/n \geqslant \sum_{\gamma} B^{|\partial\gamma|} P\left(\sum_{x\in\theta(\gamma)} \chi_x^{\Delta\gamma} = n - \Delta\gamma\right) \tag{35}$$

This yields for the Q_n the lower bound

$$Q_n = \sum_{m\geqslant n} \frac{P_m}{m} \geqslant \sum_{\gamma} B^{|\partial\gamma|} P\left(\sum_{x\in\theta(\gamma)} \chi_x^{\Delta\gamma} \geqslant n - \Delta\gamma\right) \tag{36}$$

But now the number of points x in $\theta(\gamma)$ connected to $\Delta\gamma$ is certainly larger than the number of points x connected to infinity, so we have

$$Q_n \geqslant \sum_{\gamma} B^{|\partial\gamma|} P\left(\sum_{x\in\theta(\gamma)} \chi_x^{\infty} \geqslant n - \Delta\gamma\right) \tag{37}$$

where χ_x^{∞} is the characteristic function of the event "x is connected to infinity".

Now let us give an intuitive idea of the proof. In percolative regions, we may think that the χ_x^{∞} will be sufficiently independent random variables. So, if we choose γ_n such that $\theta(\gamma_n)P_\infty \sim n$, then $P(\sum_{x\in\theta(\gamma_n)} \chi_x^{\infty} \geqslant n - \Delta\gamma_n)$ will be larger than, say, 1/2. But if $\theta(\gamma_n)P_\infty \sim n$, then $B^{|\partial\gamma|}$ is about $B'^{n^{(\nu-1)/\nu}}$, and so this term would yield the desired lower bound in (37).

We come back now to the proof, following Ref. 1 for the beginning: let us choose in $\mathbb{Z}^\nu$ the cube γ_n of side l with

$$(n/P_\infty)^{1/\nu} \leqslant l - 2 < (n/P_\infty)^{1/\nu} + 1 \tag{38}$$

Then

$$Q_n \geqslant B'^{n^{(\nu-1)/\nu}} P\left(\sum_{x\in\theta(\gamma_m)} \{\chi_x^{\infty} - \langle\chi_x^{\infty}\rangle\} \geqslant n - \Delta\gamma - (l-2)^\nu P_\infty\right) \tag{39}$$

$$\geqslant B'^{n^{(\nu-1)/\nu}} P\left(\sum_{x\in\theta(\gamma_n)} \{\chi_x^{\infty} - \langle\chi_x^{\infty}\rangle\} \geqslant -\Delta\gamma\right) \tag{40}$$

Let

$$S = \sum_{x\in\theta(\gamma_n)} \{\chi_x^{\infty} - \langle\chi_x^{\infty}\rangle\}$$

Since this variable is centered, we can apply the Bienayme–Tschebycheff inequality, which states that

$$P(|S| > a) \leqslant \langle S^2 \rangle / a^2$$

Hence

$$Q_n \geqslant B'^{n(\nu-1)/\nu}(1 - \langle S^2 \rangle / \Delta\gamma_n{}^2) \tag{41}$$

Moreover, applying FKG inequalities for the increasing functions, we have

$$\langle \chi_x{}^\infty \chi_y{}^\infty \rangle - \langle \chi_x{}^\infty \rangle \langle \chi_y{}^\infty \rangle \geqslant 0 \tag{42}$$

and then

$$\langle S^2 \rangle = \sum_{x, y \in \theta(\gamma_n)} \{\langle \chi_x{}^\infty \chi_y{}^\infty \rangle - \langle \chi_x{}^\infty \rangle \langle \chi_y{}^\infty \rangle\} \tag{43}$$

$$\leqslant (l - 2)^\nu \chi \tag{44}$$

where χ is defined by

$$\chi = \sum_{y \in \mathbb{Z}^\nu} \{\langle \chi_0{}^\infty \chi_y{}^\infty \rangle - \langle \chi_0{}^\infty \rangle \langle \chi_y{}^\infty \rangle\} \tag{45}$$

And since $\Delta\gamma \geqslant 2\nu(l - 2)^{\nu-1}$, (41) becomes

$$Q_n \geqslant B'^{n(\nu-1)/\nu}\left(1 - \frac{\chi}{4\nu^2(l - 2)^{\nu-2}}\right) \tag{46}$$

This, in the case when $\nu \geqslant 3$, ensures the desired lower bound on Q_n, if n is large enough and χ finite, and when $\nu \geqslant 2$ for any n if χ is small enough. We shall show now that χ is arbitrarily small in the three regions described in the second proposition.

Equation (45) can be rewritten as

$$\chi = \sum_{y \in \mathbb{Z}^\nu} \{\langle (1 - \chi_0{}^\infty)(1 - \chi_y{}^\infty) \rangle - \langle 1 - \chi_0{}^\infty \rangle \langle 1 - \chi_y{}^\infty \rangle\} \tag{47}$$

$$= \sum_{y \in \mathbb{Z}^\nu} \{P(E_0, E_y) - P(E_0)P(E_y)\} \tag{48}$$

where E_x is the decreasing event whose characteristic function is $(1 - \chi_x{}^\infty)$, i.e., the event "x is empty or x belongs to a finite cluster". Now if E_y occurs, let us introduce $^*C$, which is the empty *cluster (that is, the maximal empty set—possibly infinite—connected through the bonds of the *lattice) containing y if y is empty or including the external boundary of the finite cluster containing y if y is occupied. We denote by E the event "the empty *cluster $^*C$ defined above surrounds 0". Then

$$P(E_0, E_y) = P(E_0, E_y, E) + P(E_0, E_y, \bar{E}) \tag{49}$$

$$\leqslant P(E) + P(E_0, E_y, \bar{E}) \tag{50}$$

Now we shall prove that $P(E_0, E_y, \bar{E})$ is smaller than $P(E_0)P(E_y)$. If we consider a configuration such that E_y and $\bar{E}$ occur, the external boundary ∂^*C of the *cluster $^*C$ is a connected, occupied set separating 0 and Y (or possibly $0 \in \partial^*C$). Now our construction has the following consequence: the event E_0 occurs completely independently of the configuration interior to ∂^*C. (This is the reason why we have chosen an empty *cluster in the definition of the event E, and not a usual cluster; otherwise its external boundary could be a nonconnected set and E_0 would then depend on the configurations interior to that boundary.) Using this remark and since E_0 is a decreasing event and "∂^*C is occupied" an increasing event, so that the FKG inequality ensures that $P(E_0|\partial^*C) \leqslant P(E_0)$, we can get

$$P(E_0|E_y, \bar{E}) \leqslant P(E_0) \tag{51}$$

Now (51) in turn ensures that

$$P(E_0, E_y, \bar{E}) \leqslant P(E_0)P(E_y, \bar{E}) \leqslant P(E_0)P(E_y) \tag{52}$$

So (50) yields

$$P(E_0, E_y) - P(E_0)P(E_y) \leqslant P(E) \tag{53}$$

Now let us suppose that the probability to have an infinite empty *cluster is zero; then we can get an upper bound for $P(E)$ by a proof parallel to that in Proposition 2 but using now the *lattice.

So we have

$$P(E) \leqslant K''^{d(0,y)+1} \tag{54}$$

which ensures that $P(E)$ is summable over y and that the sum χ goes to zero as K'' goes to zero.

Then it remains to prove:

Lemma 2. The probability $^*\bar{P}_\infty$ that the origin belongs to an infinite empty *cluster is zero in the regions 1–3.

We know that $^*\bar{P}_\infty$ is obtained by taking the thermodynamic limit of $^*\bar{P}_\Lambda$: the probability for the origin to be connected through an empty *cluster to the boundary of the cubic box Λ centered at the origin. Clearly this probability is smaller than the probability for the origin to belong to an empty *cluster larger than $d(0, \partial\Lambda)$. Applying the same kind of calculus as for (54), we obtain

$$^*\bar{P}_\Lambda \leqslant K'^{d(0,\partial\Lambda)}$$

which goes to zero as Λ goes to infinity.

This achieves the proof of the theorem in the case of the first three regions if we choose h_0 sufficiently large and T_0 sufficiently small in order that percolation occurs and Proposition 2 holds in these regions.

4. INEQUALITIES IN THE NEGATIVE PHASE

To extend this result to the fourth region, we use an inequality derived in Ref. 3, which states that at zero magnetic field in the negative phase, the probability P_n to have a cluster of exactly n occupied sites is greater than the probability $\bar{P}_n$ to have a cluster of exactly n empty sites. Since by symmetry the inequality (4) becomes

$$\bar{P}_n/n \geqslant \exp(-\alpha' n^{(\nu-1)/\nu})$$

then

$$P_n/n \geqslant \exp(-\alpha' n^{(\nu-1)/\nu})$$

On the other hand, the upper bound is obtained similarly as in Proposition 2, using the following obvious inequality:

$$P(\Delta\mathscr{C}, \bar{\mathscr{C}} | \overline{\partial\Lambda}) \leqslant P(\Delta\mathscr{C} | \bar{\mathscr{C}}) \leqslant e^{-\beta|\mathscr{C}|}$$

which ensures the suitable upper bound and achieves the proof of the theorem.

ACKNOWLEDGMENTS

I am glad to thank B. Souillard for suggesting this study and for constant help, and M. Duneau for many useful discussions.

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